Structure theorems for certain Gorenstein ideals. *

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1 Introduction.

Let I be an ideal in the regular local ring $(R, \mathfrak{n})$ such that $I \subseteq \mathfrak{n}^2$ and let

$$A := R/I, \quad \mathfrak{m} := \mathfrak{n}/I, \quad \mathbf{k} := R/\mathfrak{n} = A/\mathfrak{m}.$$

Let $d = \dim(A)$ be the dimension, e the multiplicity and $h = v(\mathfrak{m}) - d$ the embedding codimension of A. We assume that $\mathbf{k}$ is a characteristic zero field (see the comment after Proposition 2.3).

A classical problem in the theory of local rings is the determination of the minimal number of generators $v(I) := \dim_k(I/\mathfrak{n}I)$ of the ideal I under certain restrictions on the numerical characters of A. For example, by a classical theorem of Abhyankar, we know that $e \geq h+1$, and if the equality $e = h+1$ holds we say that A has minimal multiplicity and we know that $v(I) = \binom{h+1}{2}$.

In a sequence of papers Rosales and García-Sánchez proved the following results in the case A is the one dimensional local domain corresponding to a monomial curve in the affine space, see, [4], [5], [6]. By very hard computations related to the numerical semigroup of the curve, they were able to prove that
If $h + 2 \leq e \leq h + 3$, then

$$\binom{h+2}{2} - e \leq v(I) \leq \binom{h+1}{2}. \tag{1}$$

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If $h+2 \le e \le h+4$ and A is Gorenstein, then

$$v(I) = \binom{h+1}{2} - 1. \tag{2}$$

We remark that the monomial curve $\{t^8 : t^{10} : t^{12} : t^{15}\}$ shows that (2) does not hold if $e = h+5$, see [6].

On the other hand, the monomial curve $\{t^7 : t^8 : t^{10} : t^{19}\}$ shows that the upper bound in (1) does not hold if $e = h+4$. In the same paper it is asked whether it is true that, with $e = h+4$, one has

$$\binom{h+2}{2} - e = \binom{h+1}{2} - 3 \le v(I) \le \binom{h+1}{2} + 1. \tag{3}$$

A first motivation for our paper was to understand these results and to extend them to the general case of a local Cohen-Macaulay ring of any dimension.

A sharp upper bound for the minimal number of generators of a perfect ideal I in a regular local ring R, has been given in [2] in terms of the multiplicity e and of the codimension h of R/I. The bound is

$$v(I) \le \binom{h+t-1}{t} - r + r^{<t>},$$

where the meaning of r, t and $r^{<t>}$ will be explained in the Section 2. In the same section we will also prove that

$$\binom{h+2}{2} - e \le v(I)$$

holds for every perfect codimension h ideal I in a regular local ring R, see Proposition (2.2). Further we will see how these bounds extend (1) to a considerable extent and positively answer question (3) in a very general setting.

As for (2), the problem is much harder. We have a Gorenstein local ring $(A = R/I, \mathfrak{m} = \mathfrak{n}/I)$ of codimension h and multiplicity $h+2 \le e \le h+4$ and we want to determine the minimal number of generators of I. It is easy to see that we may assume that $A = R/I$ is artinian; since A is Gorenstein, the possible Hilbert function of R/I are

$$(1, h, 1), (1, h, 1, 1), (1, h, 2, 1), (1, h, 1, 1, 1),$$

so that, in any case, $v(\mathfrak{m}^2) \le 2$.

Following Sally (see [8]), we say that an Artinian local ring $(A, \mathfrak{m})$, not necessarily Gorenstein, is **stretched** if $v(\mathfrak{m}^2) = 1$. We call **Almost stretched** an Artinian local ring such that $v(\mathfrak{m}^2) = 2$.

With this notation, we strongly extend (2) if we can prove that if R/I is Gorenstein, stretched or almost stretched of multiplicity e and codimension h, then $v(I) = \binom{h+1}{2} - 1$.

By the classical theorem of Macaulay on the shape of the Hilbert Function of a standard graded algebra, the Hilbert function of A is given by:

0	1	2	...	s	s+1
1	h	1	...	1	0

with $(s \geq 2)$ if A is stretched, or by

0	1	2	...	t	t+1	...	s	s+1
1	h	2	...	2	1	...	1	0

with $s \geq t \geq 2$, if A is almost stretched

The particular shape of the Hilbert function can be used to prove that

$$\binom{h+1}{2} - 1 \leq v(I) \leq \binom{h+1}{2} \qquad \text{if } A \text{ is stretched,}$$
$$\binom{h+1}{2} - 2 \leq v(I) \leq \binom{h+1}{2} \qquad \text{if } A \text{ is almost stretched.}$$

The case of stretched Artinian Gorenstein local ring has been studied by J. Sally in [8] where she was able to prove a structure theorem for the corresponding ideals, see also [7]. We extend this result to the case of stretched Artinian local rings of any Cohen-Macaulay type. But an unexpected and deeper result which we will prove in this paper, is a structure theorem for *any almost stretched Gorenstein local rings.*

These results are proved in Section 3 and 4, Theorem 3.1 and Theorem 4.1, respectively.

Of course, as a consequence, we get even more of what we wanted, namely:

$v(I) = \binom{h+1}{2} - 1$ if A is stretched and $\tau(A) < h$, while $v(I) = \binom{h+1}{2}$ otherwise;

$v(I) = \binom{h+1}{2} - 1$ if A is almost stretched and Gorenstein.

Another motivation for our paper came from a recent work by Casnati and Notari (see [1]). Let $\mathcal{H}ilb_{p(t)}(\mathbb{P}^n_k)$ denote the Hilbert scheme parametrizing closed subschemes in $\mathbb{P}^n_k$ with given Hilbert polynomial $p(t) \in \mathbb{Q}[t]$.

The case $\deg(p(t)) = 0$ is often problematic. Since it is known that any zero-dimensional Gorenstein scheme of degree d can be embedded as an arithmetically Gorenstein non-degenerate subscheme in $\mathbb{P}^{d-2}_k$, it is natural to study the open locus

$$\mathcal{H}ilb^{aG}_d(\mathbb{P}^{d-2}_k) \subseteq \mathcal{H}ilb_d(\mathbb{P}^{d-2}_k).$$

The scheme $\mathcal{H}ilb_d^{aG}(\mathbb{P}_k^{d-2})$ has a natural stratification which reduce the problem to understand the intrinsic structure of Artinian Gorenstein $\mathbf{k}$-algebras of degree d. Since such an algebra is the direct sum of local, Artinian, Gorenstein $\mathbf{k}$-algebras of degree at most d, it is natural to begin with the inspection of these elementary bricks.

If $d = 6$, the bricks are all given by stretched local rings, save for the case of Hilbert function $(1, 2, 2, 1)$ which is almost stretched and was studied deeply by Casnati and Notari.

If we want to extend the above results to the case $d \geq 7$, the first step is to study the intrinsic structure of Artinian Gorenstein local algebras with multiplicity 7. Since the Hilbert function $(1, 2, 3, 1)$ is not allowed, an Artinian Gorenstein ring $(A, \mathfrak{m})$ with multiplicity 7 is stretched or almost stretched. See [3] for more results on the classification of Artin algebras.

Hence, the structure theorems we will prove in the next sections will give light to these questions too.

It is clear that the best would be to have a classification up to isomorphisms of artinian Gorenstein k-algebras of a given Hilbert function, at least in the almost stretched case. We approach this very difficult problem in the last part of the paper, where we give a classification of Artinian complete intersection local k-algebras with Hilbert function $(1, 2, 2, 2, 1, 1, 1)$. This example is significant because the parameter space has a one-dimensional component.

2 Upper and lower bounds for $v(I)$.

Let $(R, \mathfrak{n})$ be a regular local ring, I an ideal in R. Let us assume that $(A = R/I, \mathfrak{m} = \mathfrak{n}/I)$ has dimension d, embedding codimension h and multiplicity e. We denote by H_A the Hilbert function of A

$$H_A(n) := dim_{\mathbf{k}} \left(\frac{\mathfrak{m}^n}{\mathfrak{m}^{n+1}} \right)$$

$n \geq 0$. The socle degree of an Artin ring A is the last integer $s = s(A)$ such that $H_A(s) \neq 0$; the Cohen-Macaulay type of A is

$$\tau(A) := dim_{\mathbf{k}}(0 : \mathfrak{m}).$$

A sharp upper bound for $v(I)$ can be given by using the notion of lex-segment ideal as in [2]. We recall that the associated graded ring of A can be presented as $gr_{\mathfrak{m}}(A) = gr_{\mathfrak{n}}(R)/I^*$, where I^* is the ideal generated by the $\mathfrak{n}$-initial forms of I in the polynomial ring $S = gr_{\mathfrak{n}}(R)$. This implies that the Hilbert Function of $A = R/I$ is the same as the Hilbert Function of the standard graded algebra S/I^*.

A set of elements in I whose $\mathfrak{n}$-initial forms generate I^*, is called a *standard basis* of I. Since it is easy to see that a standard basis is a basis, we have the inequality $v(I) \leq v(I^*)$.

On the other hand, by a classical result of Macaulay, any homogeneous ideal P in the polynomial ring $S = k[X_1, \dots, X_n]$ has the following property: the number of minimal generators of P is less than or equal to the number of minimal generators of the unique lex-segment ideal P_{lex}, which has the same Hilbert Function of P.

Hence, given the ideal I in the regular local ring $(R, \mathfrak{n})$ and the corresponding lex-segment ideal $I_{lex} := (I^*)_{lex}$ in $S := gr_{\mathfrak{n}}(R)$, we have

$$v(I) \leq v(I^*) \leq v(I_{lex}). \tag{4}$$

More difficult is to get a bound only involving the multiplicity and the codimension. Namely one has to compare the number of generators of all the lex-segment ideals having the given multiplicity and codimension. This has been done in [2] where the following bound has been proved.

We need some more notations. If n and i are positive integers then n can be uniquely written as

$$n = \binom{n(i)}{i} + \binom{n(i-1)}{i-1} + \cdots + \binom{n(j)}{j}$$

where $n(i) > n(i-1) > \cdots > n(j) \geq j \geq 1$. This is called the i-binomial expansion of n. We let

$$n^{<i>} := \binom{n(i)+1}{i+1} + \binom{n(i-1)+1}{i} + \cdots + \binom{n(j)+1}{j+1}.$$

Given two positive integers e, h with $e \geq h+1$ we define t as the unique integer such that

$$\binom{h+t-1}{t-1} \leq e < \binom{h+t}{t}$$

and

$$r := e - \binom{h+t-1}{t-1}.$$

The main result in [2] shows that, for every perfect codimension h ideal I in the regular local ring R with $I \subseteq \mathfrak{n}^2$ and $e(R/I) = e$, we have

$$v(I) \leq \binom{h+t-1}{t} - r + r^{<t>}. \tag{5}$$

For example if $h \geq 3$ and $e = h+2$, then $t = 2$, $r = 1$ and we get $v(I) \leq \binom{h+1}{2}$. The same bound holds also for $e = h+3$, see (1).

Instead, if $e = h+4$ we get $t = 2$, $r = 3$ and

$$v(I) \leq \binom{h+1}{2} - 3 + 3^{<2>} = \binom{h+1}{2} - 3 + 4 = \binom{h+1}{2} + 1,$$

see (3). The same bound holds also for $e = h+5$.

A lower bound for $v(I)$ follows from the following easy lemma.

Lemma 2.1. *Let $A = R/I$ be a local Artinian ring with multiplicity e and embedding codimension h. We assume that $I \subseteq \mathfrak{n}^2$. Then we have*

$$\binom{h+2}{2} - e \leq \binom{h+1}{2} - v(\mathfrak{m}^2) \leq v(I).$$

Proof. It is clear that the Kernel of the epimorphism

$$\mathfrak{n}^2/\mathfrak{n}^3 \to \mathfrak{m}^2/\mathfrak{m}^3 = (\mathfrak{n}^2 + I)/(\mathfrak{n}^3 + I) \to 0$$

is $(\mathfrak{n}^3 + I)/\mathfrak{n}^3 \cong I/(\mathfrak{n}^3 \cap I)$. Since $I\mathfrak{n} \subseteq \mathfrak{n}^3 \cap I$, we get

$$v(\mathfrak{n}^2) - v(\mathfrak{m}^2) = \binom{h+1}{2} - v(\mathfrak{m}^2) \leq v(I).$$

Notice that we have $e = \sum_{i=0}^{s} v(\mathfrak{m}^i)$, where s is the socle degree of A, so that $e \geq 1 + h + v(\mathfrak{m}^2)$ and

$$\binom{h+2}{2} - e \leq \binom{h+2}{2} - (1 + h + v(\mathfrak{m}^2)) = \binom{h+1}{2} - v(\mathfrak{m}^2).$$

$\square$

As a consequence of this lemma we get a lower bound for the number of generators of perfect ideals in a regular local ring which, at least for low multiplicity, seems to be useful.

Proposition 2.2. *Let Let $A = R/I$ be a local Cohen-Macaulay ring with dimension d, multiplicity e and embedding codimension h. We assume that $I \subseteq \mathfrak{n}^2$. Then we have*

$$\binom{h+2}{2} - e \leq v(I) \leq \binom{h+t-1}{t} - r + r^{<t>}.$$

Proof. Let $J = (x_1, ..., x_d)$ be a maximal $\mathfrak{n}$-superficial sequence for A. Since A is Cohen-Macaulay, $x_1, ..., x_d$ is a regular sequence modulo I so that $I \cap J = IJ$. Let

$$\overline{I} = (I + J)/J, \quad \overline{R} = R/J, \quad \overline{A} = A/(x_1, ..., x_d)A = \overline{R}/\overline{I}, \quad \overline{\mathfrak{m}} = \mathfrak{m}/J.$$

Then we have

$$v(\overline{I}) = \dim_k(I + J/\mathfrak{n}I + J) = \dim_k(I/\mathfrak{n}I + I \cap J) = \dim_k(I/\mathfrak{n}I) = v(I).$$

We know also that the multiplicity of A is the same as the multiplicity of the Artinian local ring $A/(x_1, ..., x_d)A$. Finally I and $\overline{I}$ share the same embedding codimension because $h = v(\mathfrak{m}) - d = v(\overline{\mathfrak{m}})$. The lower bound now follows from Lemma 2.1, while the upper bound is given by (5). $\square$

In the next section we are going to establish structure theorems for stretched local rings and for almost stretched Gorenstein local rings. One of the main ingredient will be the following result which will be used several times later and is reminiscent of the **lean basis** notion introduced by J. Sally in [8].

In the proof of the following Proposition we need to know that if the characteristic of $\mathbf{k}$ is 0, then **a Borel fixed monomial ideal** K **is strongly stable**. This means that K satisfies the following requirement: for any term $M \in K$ and any indeterminate X_j dividing M, we have $X_i(M/X_j) \in K$ for all $1 \leq i < j$.

Proposition 2.3. *Let* $(A, \mathfrak{m})$ *be an Artinian local ring of embedding dimension* h *and socle degree* s *such that the characteristic of the residue field* $\mathbf{k}$ *is* 0 *and* $v(\mathfrak{m}^2) \leq 2$. *Then we can find a minimal basis* $x_1, \dots, x_h$ *of* $\mathfrak{m}$ *such that*

$$\mathfrak{m}^j = (x_h^j), \qquad j = 2, \dots, s$$

if A *is stretched, while*

$$\mathfrak{m}^j = \begin{cases} (x_h^j, x_h^{j-1} x_{h-1}) & j = 2, \dots, t \\ (x_h^j) & j = t+1, \dots, s \end{cases}$$

if A *is almost stretched.*

Proof. We prove the proposition in the case A is almost stretched, because the other case is easier. Let $\mathfrak{m} = (a_1, \dots, a_h)$; we know that the Hilbert function of A is the same as the Hilbert function of $gr_{\mathfrak{m}}(A) = k[\xi_1, \dots, \xi_h] = S/J$ where $\xi_i := \overline{a_i} \in \mathfrak{m}/\mathfrak{m}^2$, $S = k[X_1, \dots, X_h]$, and J is an homogeneous ideal of S. Further, the generic initial ideal $gin(J)$ of J is a Borel fixed monomial ideal which is then strongly stable.

We claim that, after a suitable changing of coordinates in S, which corresponds to a changing of generators for the maximal ideal $\mathfrak{m}$ of A, we may assume that a basis for S_j modulo $gin(J)_j$ is given by $X_h^j, X_h^{j-1} X_{h-1}$ for $j = 2, \dots, t$, and by X_h^j for $j = t+1, \dots, s$.

In order to prove the claim, we need only to remark that if a monomial ideal K is strongly stable and $K_j \neq S_j$, then $X_h^j \notin K_j$, and if $\dim_k(S_j/K_j) \geq 2$, then also $X_h^{j-1} X_{h-1} \notin K_j$.

Since $gin(J)$ is an initial ideal, the same monomials form a basis also for S modulo J. The conclusion follows because we have for every $j \geq 0$

$$S_j/(J)_j = (\mathfrak{m}^j/\mathfrak{m}^{j+1}).$$

$\square$

Because of this Proposition, we will always assume in the paper that **the residue field k has characteristic zero**.

Remark 2.4. Notice that if the codimension is bigger than two, the argument used in the proof of Proposition 2.3 is not true anymore. Take for example the ideals (X_1^2, X_1X_2, X_1X_3) and (X_1^2, X_1X_2, X_2^2) which are strongly stable of codimension three in $k[X_1, X_2, X_3]$.

3 Stretched local rings

We recall that in [8] J. Sally studied several properties of stretched local rings and proved a structure theorem for stretched Artinian local rings in the Gorenstein case. Here we extend the result to any Cohen-Macaulay type.

Theorem 3.1. *Let I be an ideal in the regular local ring $(R, \mathfrak{n})$ such that $I \subseteq \mathfrak{n}^2$ and $A := R/I$ is Artinian. Let $\mathfrak{m} := \mathfrak{n}/I$, $h := v(\mathfrak{m})$ and τ the Cohen-Macaulay type of A.*

(1) If A is stretched of socle degree s and $\tau < h$, then we can find a basis $\{x_1, \ldots, x_h\}$ of $\mathfrak{n}$ such that I is minimally generated by the elements $\{x_i x_j\}_{1 \le i < j \le h}$, $\{x_j^2\}_{2 \le j \le \tau}$, $\{x_i^2 - u_i x_1^s\}_{\tau+1 \le i \le h}$, where the u_i are units in R.

(2) If A is stretched of socle degree s and $\tau = h$, then we can find a basis $\{x_1, x_2, \ldots, x_h\}$ of $\mathfrak{n}$ such that I is minimally generated by the elements $\{x_1 x_j\}_{2 \le j \le h}$, $\{x_i x_j\}_{2 \le i \le j \le h}$ and x_1^{s+1}.

Proof. By Proposition 2.3, we can find an element $y_1 \in \mathfrak{m}, y_1 \notin \mathfrak{m}^2$ such that $y_1^s \neq 0$ and $\mathfrak{m}^j = (y_1^j)$ for $2 \le j \le s$. We remark that this implies $y_1^j \notin \mathfrak{m}^{j+1}$ for every $1 \le j \le s$.

Lemma 3.2. *We have*

$$(0 : \mathfrak{m}) \bigcap \mathfrak{m}^2 = \mathfrak{m}^s.$$

Proof. If $s = 2$ there is nothing to prove, hence let $s \ge 3$. If $a \in 0 : \mathfrak{m}$ and $a \in \mathfrak{m}^2$, then $a = y_1^2 u$ and we get $0 = y_1 a = y_1^3 u$. Since $s \ge 3$, this implies $u \in \mathfrak{m}$, otherwise $y_1^3 = 0$. Hence $a \in \mathfrak{m}^3$; going on in this way we get $a \in \mathfrak{m}^s$ as wonted. $\square$

Since $y_1^s \in 0 : \mathfrak{m}$ and $y_1^s \neq 0$, we can find elements $y_2, \ldots, y_\tau \in \mathfrak{m}$ such that $\{y_1^s, y_2, \ldots, y_\tau\}$ is a basis of the $\mathbf{k}$-vector space $0 : \mathfrak{m}$.

Lemma 3.3. *The elements $y_1, y_2, \ldots, y_\tau$ are part of a minimal basis of $\mathfrak{m}$.*

Proof. If $\sum_{i=1}^{\tau} \lambda_i y_i \in \mathfrak{m}^2$, then $\lambda_1 \in \mathfrak{m}$, otherwise $y_1 \in 0 : \mathfrak{m} + \mathfrak{m}^2$ and $y_1^2 \in \mathfrak{m}^3$, a contradiction. Thus we get

$$\sum_{i=2}^{\tau} \lambda_i y_i \in (0 : \mathfrak{m}) \bigcap \mathfrak{m}^2 = \mathfrak{m}^s$$

and, for some $t \in R$, $\sum_{i=2}^{\tau} \lambda_i y_i + t y_1^s = 0$. This implies $\lambda_i \in \mathfrak{m}$ for every i, because $\{y_2, \ldots, y_\tau, y_1^s\}$ is a basis of the $\mathbf{k} = A/\mathfrak{m}$ vector space $0 : \mathfrak{m}$. $\square$

Of course we can complete the set $\{y_1, y_2, \ldots, y_\tau\}$ to a minimal basis of $\mathfrak{m}$, say $\mathfrak{m} = (y_1, y_2, \ldots, y_\tau, z_{\tau+1}, \ldots, z_h)$. Now, if $j \geq \tau + 1$, we have $y_1 z_j \in \mathfrak{m}^2$, hence $y_1 z_j = y_1^2 t$ and $z_j - y_1 t \in 0 : y_1$. By replacing z_j with $z_j - y_1 t$ in the minimal generators of $\mathfrak{m}$, we may assume that

$$\mathfrak{m} = (y_1, y_2, \ldots, y_\tau, y_{\tau+1}, \ldots, y_h)$$

with

$$y_2, \ldots, y_\tau \in 0 : \mathfrak{m}, \qquad y_{\tau+1}, \ldots, y_h \in 0 : y_1. \tag{6}$$

Let us first consider the case $\tau < h$.

If we choose i and j so that $\tau + 1 \leq i \leq j \leq h$, we have

$$y_i y_j \mathfrak{m} \subseteq y_i \mathfrak{m}^2 = y_i(y_1^2) = 0.$$

Hence $y_i y_j \in (0 : \mathfrak{m}) \cap \mathfrak{m}^2 = \mathfrak{m}^s = (y_1^s)$, and we can write $y_i y_j = u_{ij} y_1^s$ where $u_{ij} \in \mathfrak{m}$ if and only if $y_i y_j = 0$.

If we let $J := (y_{\tau+1}, \ldots, y_h)$, we may define an inner product in the $\mathbf{k}$-vector space $V := J/J\mathfrak{m}$ by letting

$$< \overline{y_i}, \overline{y_j} >:= \overline{u_{ij}} \in A/\mathfrak{m} = \mathbf{k}.$$

This is well defined. Namely, let $y_i = p_i + z_i$ with $p_i \in J$ and $z_i \in J\mathfrak{m}$; since $J \subseteq 0 : y_1$, we get

$$y_i y_j - p_i p_j = (p_i + z_i)(p_j + z_j) - p_i p_j \in J\mathfrak{m}^2 = y_1^2 J = 0.$$

Since the characteristic of $\mathbf{k}$ is not two, the inner product can be diagonalized. This means that the generators of $\mathfrak{m}$ can be chosen to satisfy

$$y_i y_j = 0 \tag{7}$$

for every $\tau + 1 \leq i < j \leq h$. This implies that for every $\tau + 1 \leq i \leq h$, we must have $y_i^2 \neq 0$, because, if $y_i^2 = 0$, we would get $y_i \in 0 : \mathfrak{m}$, a contradiction. Hence, for every $\tau + 1 \leq i \leq h$, we will have

$$y_i^2 = u_i y_1^s \tag{8}$$

with $u_i \notin \mathfrak{m}$.

As a consequence we can prove the first part of the theorem. Let $x_i \in \mathfrak{n}$ such that $\overline{x_i} = y_i$. From (6), (7) and (8), it is clear that all the elements

$$\{x_i x_j\}_{1 \leq i < j \leq h}, \quad \{x_j^2\}_{2 \leq j \leq \tau}, \quad \{x_i^2 - u_i x_1^s\}_{\tau+1 \leq i \leq h},$$

are in I. Let J be the ideal they generate; then $J \subseteq I$ so that $H_{R/I}(n) \leq H_{R/J}(n)$ for every $n \geq 0$. We claim that we have equality above for every $n \geq 0$. Namely we have

$$x_1^{s+1} = (u_h)^{-1} x_1 x_h^2 \in J$$

so that $I^* \supseteq J^* \supseteq K$ where K is the ideal in $S = \mathbf{k}[X_1, \dots, X_h]$ generated by X_1^{s+1} and all degree two monomials except X_1^2. Since the Hilbert function of S/K is the same as the Hilbert function of R/I, the claim follows.

From the claim we get that R/J and R/I have the same finite length so that the canonical surjection $R/J \to R/I$ is a bijection and $I = J$.

Finally, the given elements are a minimal basis of I because the generators of $\mathfrak{n}$ are analitically independent.

We come now to the case $\tau(A) = h$.

In the case the Cohen-Macaulay type of A is h, the maximum allowed, we get by (6) $\mathfrak{m} = (y_1, y_2, \dots, y_h)$ where $(y_2, \dots, y_h) \subseteq 0 : \mathfrak{m}$. This implies that $y_1 y_i = 0$ for every $i = 2, \dots, h$ and $y_i y_j = 0$ for every $2 \leq i \leq j \leq h$. Further we also have $y_1^{s+1} = 0$. The conclusion follows as in case i), but is even easier because the generators of J are monomials. $\square$

Remark 3.4. It is clear that, for a stretched local ring $A = R/I$ of maximal type, the minimal set of generators of I we have found in the above theorem are a standard basis for I. Namely we have that I^* is the ideal generated by X_1^{s+1} and the degree two monomials in S, except for X_1^2. This is not true in the case $\tau(A) < h$. In this case, the initial forms of the generators of I in $S = gr_{\mathfrak{n}}(R) = \mathbf{k}[X_1, X_2, \dots, X_h]$ are the degree two monomials in S, except for X_1^2. The ideal I^* is, as before, the ideal generated by X_1^{s+1} and the degree two monomials in S, except for X_1^2.

Remark 3.5. It is clear that, given two integers $1 \leq \tau \leq h$ and a regular local ring $(R, \mathfrak{n})$ with maximal ideal $\mathfrak{n}$ minimally generated by $(x_1, x_2, \dots, x_h)$, the ideals I generated as in Theorem 3.1 have the property that $A := R/I$ is a stretched local ring of type τ.

We have proved that if R/I is a stretched Artinian local ring of embedding dimension h, Cohen-Macaulay type $\tau < h$ and socle degree s, then we can find a minimal system of generators $x_1, \dots, x_h$ of $\mathfrak{n}$ such that

$$I = (\{x_i x_j\}_{1 \leq i < j \leq h}, \{x_j^2\}_{2 \leq j \leq \tau}, \{x_i^2 - u_i x_1^s\}_{\tau+1 \leq i \leq h})$$

where the u_i are units in R. For every $\underline{u} = (u_j)_{j=\tau+1,\dots,h}$, we let $I(\underline{u})$ such an ideal.

We will use several time the following easy and well known Lemma that is a consequence of Hensel's Lemma.

Lemma 3.6. *Let $(A, \mathfrak{m})$ be an Artinian local ring with residue field $\mathbf{k}$ and let a be an element in A such that $\overline{a} \in \mathbf{k}^*$. If $\overline{b}^n = \overline{a}$ for some $\overline{b} \in \mathbf{k}$, then $c^n = a$ for some $c \in A, c \notin \mathfrak{m}$.*

Proposition 3.7. *Let $I(\underline{u})$ as before and assume that the residue field $\mathbf{k} = R/\mathfrak{n}$ verifies $\mathbf{k}^{1/2} \subseteq \mathbf{k}$. Then we can find a system of generators $y_1, \dots, y_h$ of $\mathfrak{n}$ such that*

$$I(\underline{u}) = (\{y_i y_j\}_{1 \le i < j \le h}, \{y_j^2\}_{2 \le j \le \tau}, \{y_i^2 - y_1^s\}_{\tau+1 \le i \le h}).$$

Proof. Since $\mathbf{k}^{1/2} \subseteq \mathbf{k}$, by the above Lemma we can find, for every $i = \tau+1, \dots, h$, elements $v_i \in R$ such that $v_i^2 \cong 1/u_i \bmod I(\underline{u})$. Hence $v_i \notin \mathfrak{n}$ and we get

$$v_i^2 x_i^2 - x_1^s \cong (1/u_i)x_i^2 - x_1^s = (1/u_i)(x_i^2 - u_i x_1^s) \cong 0.$$

This proves that if we let

$$y_i = x_i, \quad \text{for} \quad i = 1, \dots, \tau, \quad y_i = v_i x_i \quad \text{for} \quad i = \tau+1, \dots, h,$$

then

$$(\{y_i y_j\}_{1 \le i < j \le h}, \{y_j^2\}_{2 \le j \le \tau}, \{y_i^2 - x_1^s\}_{\tau+1 \le i \le h}) \subseteq I(\underline{u}).$$

Since the two ideals have the same Hilbert function, they must coincide. □

4 Almost stretched Gorenstein local rings

In this section we are considering Artinian local rings $(A, \mathfrak{m})$ such that the square of the maximal ideal is minimally generated by two elements. Recall that in Section 1 such a ring A has been called almost stretched. If A is almost stretched and Gorenstein, the Hilbert function of A is given by

0	1	2	...	t	t+1	...	s	s+1
1	h	2	...	2	1	...	1	0

with $h \ge 2$ and $s \ge t+1 \ge 3$.

The structure result for almost stretched Gorenstein local rings will be a consequence of the following theorem.

Theorem 4.1. *Let $(A, \mathfrak{m})$ be an Artinian local ring which is Gorenstein with embedding dimension h. If A is almost stretched, then we can find integers $s \ge t+1 \ge 3$ and a minimal basis $x_1, \dots, x_h$ of $\mathfrak{m}$ such that*

$$\begin{cases} x_1 x_j = 0 & \text{for } j = 3, \dots, h \\ x_i x_j = 0 & \text{for } 2 \le i < j \le h \\ x_j^2 = u_j x_1^s & \text{for } j = 3, \dots, h \\ x_2^2 = a x_1 x_2 + w x_1^{s-t+1} & \\ x_1^t x_2 = 0 & \end{cases}$$

with suitable $w, u_3, \dots, u_h \notin \mathfrak{m}$ and $a \in A$.

Proof. By Proposition 2.3 we may assume that $\mathfrak{m} = (x_1, \ldots, x_h)$ with

$$\mathfrak{m}^j = \begin{cases} (x_1^j, x_1^{j-1}x_2) & j = 2, \ldots, t \\ (x_1^j) & j = t+1, \ldots, s. \end{cases}$$

We claim that we may assume also $(x_3, \ldots, x_h) \subseteq (0) : x_1$. Namely, for $j \geq 3$, we can write $x_1x_j = b_jx_1^2 + c_jx_1x_2$, hence $x_1(x_j - b_jx_1 - c_jx_2) = 0$. We get the claim by replacing x_j with $x_j - b_jx_1 - c_jx_2$ for every $j \geq 3$. This means that we have

$$x_1x_3 = x_1x_4 = \cdots = x_1x_h = 0. \tag{9}$$

Further, since $\mathfrak{m}^{t+1} = (x_1^{t+1})$, for some $c \in A$, we have

$$x_1^t x_2 = cx_1^{t+1}. \tag{10}$$

Let $y_2 := x_2 - cx_1$, then

$$x_1^t y_2 = x_1^t(x_2 - cx_1) = x_1^t x_2 - cx_1^{t+1} = 0.$$

Since x_2 is not involved in equations (9), we may replace x_2 with y_2 in the generating set of $\mathfrak{m}$. Hence we may assume that

$$x_1^t x_2 = 0. \tag{11}$$

We notice that $x_1^{t-1}x_2 \notin \mathfrak{m}^s$, otherwise $x_1^{t-1}x_2 \in \mathfrak{m}^{t+1}$, a contradiction to the fact that $x_1^{t-1}x_2, x_1^t$ is a minimal basis of $\mathfrak{m}^t$. This implies that $x_1^{t-1}x_2$ cannot be in the socle of A. Since by (11) and (9)

$$x_1^{t-1}x_2 \in (0) : (x_1, x_3, \ldots, x_h),$$

we must have

$$x_1^{t-1}x_2^2 \neq 0. \tag{12}$$

We want to prove now that we can find $a \in A$, $w \notin \mathfrak{m}$ such that

$$x_2^2 = ax_1x_2 + wx_1^{s-t+1}.$$

In order to prove this we need the following easy remarks.

Claim 1. If for some $r, p \in A$ and $n \geq 2$ we have $x_2^2 = rx_1x_2 + px_1^n$, then $n \leq s - t + 1$. If further $p \notin \mathfrak{m}$, then $n = s - t + 1$.

Proof of Claim 1. We have

$$x_1^{t-1}x_2^2 = x_1^{t-1}(rx_1x_2 + px_1^n) = px_1^{n+t-1}$$

because by (11) $x_1^t x_2 = 0$. Since by (12) $x_1^{t-1}x_2^2 \neq 0$, this implies $n + t - 1 \leq s$. We have also $px_1^n = x_2(x_2 - rx_1)$, hence, if $p \notin \mathfrak{m}$, $x_1^n = vx_2$ for some $v \in A$. As a

consequence we get $x_1^{n+t} = vx_1^t x_2 = 0$. Since $x_1^s \neq 0$, we have $n + t \geq s + 1$ and the conclusion follows.

Claim 2. If for some $n \geq 2$, $a \in A$ and $b \in \mathfrak{m}$, we have $x_2^2 = ax_1x_2 + bx_1^n$ then for some $c, d \in A$ we have $x_2^2 = cx_1x_2 + dx_1^{n+1}$.

Proof of Claim 2 . This is easy because by (9) $x_1x_j = 0$ for every $j \geq 3$.

Claim 3. If for some $a, b \in A$ we have $x_2^2 = ax_1x_2 + bx_1^{s-t+1}$ then $b \notin \mathfrak{m}$.

Proof of Claim 3. If, by contradiction, $b \in \mathfrak{m}$, then by Claim 2 and 1 we get

$$s - t + 2 \leq s - t + 1.$$

Since $\mathfrak{m}^2 = (x_1^2, x_1x_2)$, we have $x_2^2 = ax_1x_2 + bx_1^2$ for some $a, b \in A$. Thus, as a trivial consequence of these three claims, we get that for some $a \in A$ and $w \notin \mathfrak{m}$

$$x_2^2 = ax_1x_2 + wx_1^{s-t+1}. \tag{13}$$

Now we recall that for every $j \geq 3$, we have by (9)

$$x_j\mathfrak{m}^2 = x_j(x_1^2, x_1x_2) = 0,$$

so that, by using the Gorenstein assumption, we get

$$x_j\mathfrak{m} \subseteq (0) : \mathfrak{m} = (x_1^s). \tag{14}$$

Let us consider the ideal $J := (x_3, \ldots, x_h)$. By (14), for every $3 \leq i \leq j \leq h$, we have $x_ix_j = u_{ij}x_1^s$ with $u_{ij} \in A$. We notice that if we have also $x_ix_j = w_{ij}x_1^s$, then $(u_{ij} - w_{ij})x_1^s = 0$, which implies $u_{ij} - w_{ij} \in \mathfrak{m}$.

Hence we may define an inner product in the $\mathbf{k} = A/\mathfrak{m}$-vector space $V := J/J\mathfrak{m}$ by letting

$$< \overline{x_i}, \overline{x_j} >:= \overline{u_{ij}} \in A/\mathfrak{m}$$

and extending this definition by bilinearity to $V \times V$.

Since the characteristic of $\mathbf{k}$ is not two, the inner product can be diagonalized. This means that we can find minimal generators $y_3, \ldots, y_h$ of J such that $y_iy_j = 0$ for $i \neq j$. If we replace $x_3, \ldots, x_h$ with $y_3, \ldots, y_h$ in the generating set of $\mathfrak{m}$, it is clear that equations (9), (11), (13) and (14) are still valid. Thus generators $x_1, \ldots, x_h$ of $\mathfrak{m}$ can be chosen so that

$$x_ix_j = 0 \tag{15}$$

for every i and j such that $3 \leq i < j \leq h$.

From (14) and for every $j \geq 3$ we have

$$x_j^2 = u_jx_1^s$$

with $u_j \in A$. We claim that $u_j \notin \mathfrak{m}$ for every $j \geq 3$.

In order to prove this claim, let us remember that again by (14) we have

$$x_2x_j = a_jx_1^s$$

for every $j \geq 3$ and suitable $a_j \in A$. We fix $j \geq 3$ and let

$$\rho := wx_j - a_jx_1^{t-1}x_2.$$

Since $w \notin \mathfrak{m}$, it is clear that $\rho \notin \mathfrak{m}^2$ so that $\rho \notin \mathfrak{m}^s \subseteq \mathfrak{m}^2$. This implies that ρ **cannot be in the socle of** A. We will use the following equalities:

$x_1x_j = 0$	for $j \geq 3$	see (9)
$x_1^tx_2 = 0$		see (11)
$x_2^2 = ax_1x_2 + wx_1^{s-t+1}$		see (13)
$x_jx_k = 0$	for $3 \leq j < k \leq h$	see (15)

We have

$\rho x_1 = wx_1x_j - a_jx_1^tx_2 = 0,$

$\rho x_2 = wx_2x_j - a_jx_1^{t-1}x_2^2 = wa_jx_1^s - a_jx_1^{t-1}(ax_1x_2 + wx_1^{s-t+1}) = wa_jx_1^s - wa_jx_1^s = 0,$

$\rho x_k = wx_jx_k - a_jx_1^{t-1}x_2x_k = 0 \quad$ if $k \geq 3$, $k \neq j$,

$\rho x_j = wx_j^2 - a_jx_1^{t-1}x_2x_j = wu_jx_1^s.$

Since ρ cannot be in the socle, we must have $u_j \notin \mathfrak{m}$. This proves the Claim.

As a consequence we may assume that for every $j \geq 3$ and suitable $u_j \notin \mathfrak{m}$ we have

$$x_j^2 = u_jx_1^s. \tag{16}$$

We come now to the last manipulation of our elements. As a consequence of the above claim, we may consider the element

$$y_2 := x_2 - \sum_{i=3}^{h} u_i^{-1}a_ix_i.$$

For every $j = 3, \ldots, h$ we have by using (15)

$$y_2x_j = x_2x_j - \sum_{i=3}^{h} u_i^{-1}a_ix_ix_j = a_jx_1^s - u_j^{-1}a_jx_j^2 = a_jx_1^s - u_j^{-1}a_ju_jx_1^s = 0.$$

Further we have

$$x_1^tx_2 = x_1^t(y_2 + \sum_{i=3}^{h} u_i^{-1}a_ix_i) = x_1^ty_2.$$

Finally let $d := x_2 - y_2 = \sum_{i=3}^h u_i^{-1} a_i x_i$. Then $d \in J := (x_3, \ldots, x_h)$ and we have

$$x_1 d = 0 \qquad y_2 d = 0.$$

Since, by (14), $J\mathfrak{m} \subseteq (x_1^s)$, we have

$$d^2 = p x_1^s$$

for some $p \in A$. It follows that

$$x_2^2 - a x_1 x_2 - w x_1^{s-t+1} = (y_2+d)^2 - a x_1 (y_2 + d) - w x_1^{s-t+1} = y_2^2 + d^2 - a x_1 y_2 - w x_1^{s-t+1} =$$

$$= y_2^2 - a x_1 y_2 - w x_1^{s-t+1} + p x_1^s = y_2^2 - a x_1 y_2 - (w - p x_1^{t-1}) x_1^{s-t+1}$$

where $w - p x_1^{t-1} \notin \mathfrak{m}$.

Thus we may replace x_2 with y_2 and finally we get a basis $x_1, \ldots, x_h$ for $\mathfrak{m}$ so that

$$\begin{cases} x_1 x_j = 0 & \text{for } j = 3, \ldots, h \\ x_i x_j = 0 & \text{for } 2 \le i < j \le h \\ x_j^2 = u_j x_1^s & \text{for } j = 3, \ldots, h \\ x_2^2 = a x_1 x_2 + w x_1^{s-t+1} & \\ x_1^t x_2 = 0 & \end{cases}$$

with suitable $w, u_3, \ldots, u_h \notin \mathfrak{m}$ and $a \in A$. □

As a consequence of this theorem we get a structure theorem for almost stretched Artinian and Gorenstein local rings.

Corollary 4.2. *Let $(R, \mathfrak{n})$ be a regular local ring of dimension h and $I \subseteq \mathfrak{n}^2$ an ideal such that $(A = R/I, \mathfrak{m} = \mathfrak{n}/I)$ is almost stretched Artinian and Gorenstein. Then there is a minimal basis $x_1, \ldots, x_h$ of $\mathfrak{n}$ such that I is minimally generated by the elements*

$$\{x_1 x_j\}_{j=3,\ldots,h} \quad \{x_i x_j\}_{2 \le i < j \le h} \quad \{x_j^2 - u_j x_1^s\}_{j=3,\ldots,h} \quad x_2^2 - a x_1 x_2 - w x_1^{s-t+1}, \qquad x_1^t x_2.$$

with $w, u_3, \ldots, u_h \notin \mathfrak{n}$ and $a \in R$.

Proof. By the Theorem 4.1 we can find a basis $x_1, \ldots, x_h$ of $\mathfrak{n}$ such that the ideal J generated by the above elements is contained in I. We need to show that I is indeed equal to J. We first remark that modulo J we have

$$x_1^{s+1} = x_1^t x_1^{s-t+1} \cong x_1^t \frac{x_2^2 - a x_1 x_2}{w} \cong x_1^t x_2 \frac{x_2 - a x_1}{w} \cong 0$$

so that $x_1^{s+1} \in J$.

Passing to the ideals of initial forms in the polynomial ring

$$S = gr_{\mathfrak{n}}(R) = \oplus_{j\geq 0}(\mathfrak{n}^j/\mathfrak{n}^{j+1}) = (R/\mathfrak{n})[X_1,\dots,X_h],$$

we have

$$I^* \supseteq J^* \supseteq K$$

where K is the ideal in S generated by the elements

$$\{X_1X_j\}_{j=3,\dots,h} \quad \{X_iX_j\}_{2\leq i<j\leq h} \quad \{X_j^2\}_{j=3,\dots,h} \quad X_1^tX_2, \quad X_1^{s+1}$$

and the quadric $Q := X_2^2 - \overline{a}X_1X_2$ in the case $s \geq t+2$, or $Q := X_2^2 - \overline{a}X_1X_2 - \overline{w}X_1^2$ in the case $s = t+1$.

In both cases we have $X_jS_1 \subseteq K$ for every $j \geq 3$ so that

$$(K + (X_3,\dots,X_h))_n = K_n$$

for every $n \neq 1$. This implies that for every $n \neq 1$

$$H_{S/K}(n) = H_{S/(K+(X_3,\dots,X_h))}(n) = H_{\mathbf{k}[X_1,X_2]/(Q,X_1^tX_2,X_1^{s+1})}(n).$$

Now we compute the Hilbert Function of the graded algebra $\mathbf{k}[X_1,X_2]/(Q,X_1^tX_2,X_1^{s+1})$. We let $B := \mathbf{k}[X_1,X_2]$; in the case $Q = X_2^2 - \overline{a}X_1X_2 = X_2(X_2 - \overline{a}X_1)$, we have an exact sequence

$$0 \to B/(X_2 - \overline{a}X_1, X_1^t)(-1) \xrightarrow{X_2} B/(Q, X_1^tX_2) \to B/(X_2) \to 0$$

which enables us to compute the Hilbert Series of $B/(Q, X_1^tX_2)$:

$$P_{B/(Q,X_1^tX_2)}(z) = zP_{B/(X_2-\overline{a}X_1,X_1^t)}(z) + P_{B/(X_2)}(z) =$$

$$= \frac{z(1-z)(1-z^t) + (1-z)}{(1-z)^2} = \frac{1+z-z^{t+1}}{1-z}$$

which gives the Hilbert Function

0	1	2	...	t	t+1	...	s	s+1	s+2	...
1	2	2	...	2	1	...	1	1	1	...

Since $X_1^{s+1} \notin (Q, X_1^tX_2)$, the Hilbert Function of $\mathbf{k}[X_1,X_2]/(Q,X_1^tX_2,X_1^{s+1})$ is

0	1	2	...	t	t+1	t+2	...	s	s+1
1	2	2	...	2	1	1	...	1	0

so that the Hilbert Function of S/K is

0	1	2	...	t	t+1	t+2	...	s	s+1
1	h	2	...	2	1	1	...	1	0

the same as that of S/I^*.

In the case $s = t+1$ we have $Q = X_2^2 - \overline{a}X_1X_2 - \overline{w}X_1^2$ with $\overline{w} \neq 0$. Hence $\{Q, X_1^tX_2\}$ is a regular sequence and $\mathbf{k}[X_1, X_2]/(Q, X_1^tX_2)$ has Hilbert Function

0	1	2	...	t	t+1=s	t+2
1	2	2	...	2	1	0

We remark that in this case we have $X_1^2 \in (Q, X_2)$ so that

$$X_1^{s+1} = X_1^{t+2} = X_1^tX_1^2 \in (Q, X_1^tX_2).$$

In any case we have proven that S/I^* and S/K have the same Hilbert Function. This implies that $I^* = J^* = K$ so that the Hilbert Function of R/I and R/J are the same. Hence R/I and R/J have the same finite length, so the canonical epimorphism $R/J \to R/I$ is an isomorphism and $I = J$ as claimed. □

Remark 4.3. Notice that in the proof of Corollary 4.2 we describe the ideal I^*: is generated by

$$\{X_1X_j\}_{j=3,\ldots,h} \;\; \{X_iX_j\}_{2\leq i<j\leq h} \;\; \{X_j^2\}_{j=3,\ldots,h} \;\; X_1^tX_2, \quad X_1^{s+1}$$

and the quadric $Q := X_2^2 - \overline{a}X_1X_2$ in the case $s \geq t+2$, or $Q := X_2^2 - \overline{a}X_1X_2 - \overline{w}X_1^2$ in the case $s = t+1$, with $\overline{w} \neq 0, \overline{a} \in \mathbf{k}$.

We want to prove now the converse of the above result. Notice that for the next Lemma we even do not need neither regular nor local.

Lemma 4.4. *Let B a ring, $t \geq 2$, $h \geq 2$, $s \geq t+1$ and $\mathfrak{n} = (x_1, \ldots, x_h)$ an ideal in B. Let J be the ideal generated by*

$$\{x_1x_j\}_{j=3,\ldots,h} \;\; \{x_ix_j\}_{2\leq i<j\leq h} \;\; \{x_j^2 - u_jx_1^s\}_{j=3,\ldots,h} \;\; x_2^2 - ax_1x_2 - wx_1^{s-t+1}, \quad x_1^tx_2.$$

If w is a unit in B, then

$$\mathfrak{n}^{s+1} \subseteq J.$$

Proof. For every $i \neq j$, save for $(i,j) = (1,2)$, we have

$$x_ix_j \in J.$$

For every $3 \leq j \leq h$,

$$x_j^2 \in J + (x_1^s),$$

and since $s - t + 1 \geq 2$,

$$x_2^2 \in J + (x_1^2, x_1x_2).$$

We claim that for every $r \geq 2$ we have

$$\mathfrak{n}^r \subseteq J + (x_1^r, x_1^{r-1}x_2).$$

If $r = 2$, we have $\mathfrak{n}^2 \subseteq J + (x_1^2, x_1x_2)$ by the above three properties. Let us proceed by induction on r. We have

$$\mathfrak{n}^{r+1} = \mathfrak{n}\mathfrak{n}^r \subseteq J + \mathfrak{n}(x_1^r, x_1^{r-1}x_2) =$$

$$= J + (x_1, x_2)(x_1^r, x_1^{r-1}x_2) = J + (x_1^{r+1}, x_1^r x_2, x_1^{r-1}x_2^2).$$

The claim follows because $x_2^2 \in J + (x_1^2, x_1x_2)$, so

$$x_1^{r-1}x_2^2 \in J + (x_1^{r+1}, x_1^r x_2).$$

From the claim we have $\mathfrak{n}^{s+1} \subseteq J + (x_1^{s+1}, x_1^s x_2)$. Since $s \geq t$, we get $x_1^s x_2 \in (x_1^t x_2) \subseteq J$; on the other hand, since w is a unit we get modulo J the equalities

$$x_1^{s+1} = (x_1^t/w)wx_1^{s-t+1} \cong (x_1^t/w)(x_2^2 - ax_1x_2) \cong 0.$$

The conclusion follows. $\square$

We come now to a very crucial step in our way.

Lemma 4.5. *Let R be a regular local ring of dimension $h \geq 2$, $\mathfrak{n} = (x_1, \dots, x_h)$ the maximal ideal of R, $s \geq t + 1 \geq 3$ and $a, u_3, \dots, u_h, w \in R$. Let I be the ideal generated by*

$$\{x_1x_j\}_{j=3,\dots,h} \quad \{x_ix_j\}_{2 \leq i < j \leq h} \quad \{x_j^2 - u_jx_1^s\}_{j=3,\dots,h} \quad q := x_2^2 - ax_1x_2 - wx_1^{s-t+1}, x_1^t x_2.$$

If $u_3, \dots, u_h, w \notin \mathfrak{n}$, then

(1) $\overline{x_1}^t, \overline{x_1}^{t-1}\overline{x_2} \in (\mathfrak{n}^t + I)/(\mathfrak{n}^{t+1} + I)$ *are $(R/\mathfrak{n})$-linearly independent elements,*

(2) $x_1^s \notin I$.

Proof. In order to prove (1) we need to show that if $\lambda x_1^t + \mu x_1^{t-1}x_2 \in I + \mathfrak{n}^{t+1}$, then $\lambda, \mu \in \mathfrak{n}$. It is clear that if $\lambda x_1^t + \mu x_1^{t-1}x_2 \in I + \mathfrak{n}^{t+1}$, then

$$\lambda x_1^t + \mu x_1^{t-1}x_2 \in I + \mathfrak{n}^{t+1} + (x_3, \dots, x_h) = (x_3, \dots, x_h) + (x_1, x_2)^{t+1} + (x_1^s, x_1^t x_2, q)$$

$$= (x_3, \dots, x_h) + (x_1, x_2)^{t+1} + (q).$$

Let's read the above condition in the two dimensional regular local ring $T := R/(x_3, \dots, x_h)$, whose maximal ideal is generated by the residue class of x_1 and x_2 modulo $(x_3, \dots, x_h)$. By abuse of notation, we again denote these elements by x_1, x_2 and the maximal ideal of T by $\mathfrak{n}$. We have

$$\lambda x_1^t + \mu x_1^{t-1}x_2 = eq + z$$

where $z \in \mathfrak{n}^{t+1}$. This implies that $eq \in \mathfrak{n}^t$. If $eq \in \mathfrak{n}^{t+1}$, the conclusion follows by the analytic independence of x_1, x_2. If $eq \notin \mathfrak{n}^{t+1}$, then since $q = x_2^2 - ax_1x_2 - wx_1^{s-t+1} \in \mathfrak{n}^2$,

we have $e \in \mathfrak{n}^{t-2}$, $e \notin \mathfrak{n}^{t-1}$. By passing to the associated graded ring $(T/\mathfrak{n})[X_1, X_2]$ of T, we get

$$X_1^{t-1}(\overline{\lambda} X_1 + \overline{\mu} X_2) = e^* q^*.$$

Since X_1 is not a factor of q^*, X_1^{t-1} must be a factor of e^*. This is a contradiction because e^* is an homogeneous element of degree $t-2$. The conclusion follows.

Let us prove (2). By contradiction, let

$$x_1^s = \sum_{j=3}^{h} \lambda_j x_1 x_j + \sum_{j=3}^{h} \rho_j (x_j^2 - u_j x_1^s) + \sum_{2 \le i < j \le h} \mu_{ij} x_i x_j + \sigma x_1^t x_2 + \alpha q.$$

Since $s \ge t+1 \ge 3$, this implies

$$\sum_{j=3}^{h} \lambda_j x_1 x_j + \sum_{j=3}^{h} \rho_j x_j^2 + \sum_{2 \le i < j \le h} \mu_{ij} x_i x_j + \alpha(x_2^2 - a x_1 x_2 - w x_1^{s-t+1}) \in \mathfrak{n}^3.$$

By the analytic independence of $x_1, \ldots, x_h$, all the coefficients of the degree two monomials in $x_1, \ldots, x_h$ must be in $\mathfrak{n}$. In particular $\rho_j \in \mathfrak{n}$ for every $j = 1, \ldots, h$. This implies that

$$x_1^s \in (x_3, \ldots, x_h) + (x_1^t x_2, q) + \mathfrak{n}^{s+1}.$$

As we did before, we pass to the two dimensional regular local ring $T := R/(x_3, \ldots, x_h)$ whose maximal ideal is still denoted by $\mathfrak{n}$ and generated by x_1, x_2. We can write

$$x_1^s = \sigma x_1^t x_2 + \alpha q + \beta \tag{17}$$

where $\beta \in \mathfrak{n}^{s+1}$. This implies that $x_1^s + \alpha w x_1^{s-t+1} \in (x_2, x_1^{s+1})$ so that we can write $x_1^s + \alpha w x_1^{s-t+1} = x_2 a + x_1^{s+1} b$ for some $a, b \in T$. This gives

$$x_1^{s-t+1}(x_1^{t-1} + \alpha w - b x_1^t) = x_2 a.$$

Since x_1^{s-t+1}, x_2 is a regular sequence in T, we get $x_1^{t-1} + \alpha w - b x_1^t = x_2 c$ for some $c \in T$. Hence $\alpha w = x_1^{t-1}(b x_1 - 1) + x_2 c$ and since w is a unit, we finally get

$$\alpha = v x_1^{t-1} + d x_2$$

for some $v, d \in T$, $v \notin \mathfrak{n}$. Let us use this formula in equation (17). We get

$$x_1^s = \sigma x_1^t x_2 + (v x_1^{t-1} + d x_2) q + \beta \tag{18}$$

where $\beta \in \mathfrak{n}^{s+1}$ and $v \notin \mathfrak{n}$.

We claim now that if for some $r \ge 2$ and $j \ge 2$ we have, as in (18) with $j = s$ and $r = t$,

$$x_1^j - \sigma x_1^r x_2 - (v x_1^{r-1} + d x_2) q \in \mathfrak{n}^{j+1}$$

then, for suitable $e \in T$, we get also

$$x_1^{j-1} - \sigma x_1^{r-1} x_2 - (v x_1^{r-2} + e x_2) q \in \mathfrak{n}^j.$$

Since $q = x_2^2 - ax_1x_2 - wx_1^{s-t+1}$, the assumption of the claim implies

$$dx_2^3 \in (x_1) + \mathfrak{n}^{j+1} = (x_1) + (x_2^{j+1}).$$

Now, since $j+1 \geq 3$ and x_1, x_2^3 is a regular sequence, we get $d = ex_1 + fx_2^{j-2}$ for some $e, f \in T$ so that $x_1^j - \sigma x_1^r x_2 - (vx_1^{r-1} + ex_1x_2)q \in \mathfrak{n}^{j+1}$. Since $\mathfrak{n}^{j+1} \cap (x_1) = x_1\mathfrak{n}^j$, it follows that

$$x_1^{j-1} - \sigma x_1^{r-1}x_2 - (vx_1^{r-2} + ex_2)q \in \mathfrak{n}^j$$

and the claim is proved.

Starting from (18), where we let $j = s$ and $r = t$, we apply $t-1$ times the claim and we get

$$x_1^{s-t+1} - \sigma x_1x_2 - (v + gx_2)q \in \mathfrak{n}^{s-t+2}$$

for some $g \in T$. This implies

$$(v + gx_2)x_2^2 \in (x_1) + \mathfrak{n}^{s-t+2} = (x_1, x_2^{s-t+2}),$$

so that, since $s - t + 2 \geq 3$, we get $vx_2^2 \in (x_1, x_2^3)$, which is a contradiction because $v \notin \mathfrak{n}$. $\square$

Corollary 4.6. *Let R be a regular local ring of dimension $h \geq 2$, $\mathfrak{n} = (x_1, \ldots, x_h)$ the maximal ideal of R, $s \geq t+1 \geq 3$ and $a, u_3, \ldots, u_h, w \in R$. Let I be the ideal generated by*

$$\{x_1x_j\}_{j=3,\ldots,h} \quad \{x_ix_j\}_{2\leq i<j\leq h} \quad \{x_j^2 - u_jx_1^s\}_{j=3,\ldots,h} \quad q := x_2^2 - ax_1x_2 - wx_1^{s-t+1}, x_1^tx_2.$$

If $u_3, \ldots, u_h, w \notin \mathfrak{n}$, then the Hilbert Function of R/I is

0	1	2	...	t	t+1	t+2	...	s	s+1
1	h	2	...	2	1	1	...	1	0

Proof. We have seen in the proof of Lemma 4.4 that $\mathfrak{n}^r \subseteq J + (x_1^r, x_1^{r-1}x_2)$ for every $r \geq 2$. This proves that all the powers of $\mathfrak{n}/I$ can be generated by two elements. By a) of Lemma 4.5 we get $H_{R/I}(t) = 2$, which implies, by the characterization of Hilbert functions due to Macaulay, $H_{R/I}(j) = 2$ for every $2 \leq j \leq t$. Since $x_1^tx_2 \in I$, we also have $H_{R/I}(t+1) \leq 1$, which implies $H_{R/I}(j) \leq 1$ for every $j \geq t+1$. The conclusion follows because $x_1^s \notin I$ and $\mathfrak{n}^{s+1} \subseteq I$. $\square$

We are ready to prove the converse of Corollary 4.2.

Theorem 4.7. *Let R be a regular local ring of dimension $h \geq 2$, $\mathfrak{n} = (x_1, \dots, x_h)$ the maximal ideal of R, $s \geq t+1 \geq 3$ and $a, u_3, \dots, u_h, w \in R$. Let I be the ideal generated by*

$$\{x_1x_j\}_{j=3,\dots,h} \quad \{x_ix_j\}_{2\leq i<j\leq h} \quad \{x_j^2 - u_jx_1^s\}_{j=3,\dots,h} \quad x_2^2 - ax_1x_2 - wx_1^{s-t+1}, x_1^tx_2.$$

If $u_3, \dots, u_h, w \notin \mathfrak{n}$, then R/I is an almost stretched Gorenstein local ring with Hilbert function

0	1	2	...	t	t+1	t+2	...	s	s+1
1	h	2	...	2	1	1	...	1	0

Proof. After the above Corollary we need only to prove that R/I is Gorenstein.

We let $\mathfrak{m} := \mathfrak{n}/I$ and $y_i := \overline{x_i} \in A = R/I$. By Lemma 4.5 we have $\mathfrak{m}^j = (y_1^j, y_1^{j-1}y_2)$ for every $j = 2, \dots t$, and $\mathfrak{m}^j = (y_1^j)$ for $j = t+1, \dots, s$. We prove the theorem in three steps.

Claim 1. If for some $j \neq 1, t, s$ and some $r \in \mathfrak{m}^j$ we have $ry_1 = 0$, then $r \in \mathfrak{m}^{j+1}$.

Proof of Claim 1. Let $2 \leq j \leq t-1$; then $r = \lambda y_1^j + \mu y_1^{j-1}y_2$. We have

$$0 = ry_1 = \lambda y_1^{j+1} + \mu y_1^j y_2.$$

Since $y_1^{j+1}, y_1^jy_2$ is a minimal basis of $\mathfrak{m}^{j+1}$, we have $\lambda, \mu \in \mathfrak{m}$ and $r \in \mathfrak{m}^{j+1}$. The case $t+1 \leq j \leq s-1$ is even easier.

Claim 2. If for some $r \in \mathfrak{m}^t$ we have $ry_1 = ry_2 = 0$, then $r \in \mathfrak{m}^{t+1}$.

Proof of Claim 2. Let $r = \lambda y_1^t + \mu y_1^{t-1}y_2$. Since $y_1^t y_2 = 0$, we have $0 = ry_1 = \lambda y_1^{t+1}$. This implies $\lambda \in \mathfrak{m}$. On the other hand we have

$$0 = ry_2 = \mu y_1^{t-1}y_2^2 = \mu y_1^{t-1}(\overline{a}y_1y_2 + \overline{w}y_1^{s-t+1}) = \mu\overline{w}y_1^s.$$

Since $\overline{w}$ is a unit in A, this implies $0 = \mu y_1^s$ so that $\mu \in \mathfrak{m}$. Thus $r \in \mathfrak{m}^{t+1}$.

These two Claims prove that if $r \in \mathfrak{m}^2$ and $ry_1 = ry_2 = 0$, then $r \in \mathfrak{m}^s$.

Claim 3. If $r \in (0) : \mathfrak{m}$ then $r \in \mathfrak{m}^2$, so that $r \in \mathfrak{m}^s$ and A is Gorenstein.

Proof of Claim 3. Let $r \in (0) : \mathfrak{m}$; then $r \in \mathfrak{m}$ and we can write $r = \sum_{i=1}^h \lambda_i y_i$. Since $y_1y_j = 0$ for every $j \geq 3$, we have

$$0 = ry_1 = \lambda_1 y_1^2 + \lambda_2 y_1 y_2.$$

This implies $\lambda_1, \lambda_2 \in \mathfrak{m}$ so that $r = \sum_{i=3}^h \lambda_i y_i + b$ with $b \in \mathfrak{m}^2$.

Since $y_2y_j = 0$ for every $j \geq 3$, we get $0 = ry_1 = by_1$ $0 = ry_2 = by_2$; by Claim 2 this implies $b \in \mathfrak{m}^s$. Since $y_iy_j = 0$ for every $3 \leq i < j \leq h$, and $\mathfrak{m}^{s+1} = 0$, we get

$$0 = ry_j = \lambda_j y_j^2 = \lambda_j \overline{u_j} y_1^s.$$

Since $\overline{u_j}$ is a unit in A, this implies $\lambda_j y_1^s = 0$ so that $\lambda_j \in \mathfrak{m}$ and $r \in \mathfrak{m}^2$. The proof of the Claim 3 and of the theorem is complete. $\square$

The structure theorem of almost stretched Gorenstein local rings we have proved, can be refined under a mild assumption on the residue field of R. This will be crucial for the study of the moduli problem and it is a consequence of the main structure Theorem 4.1 and Lemma 3.6.

Proposition 4.8. *Let $(R, \mathfrak{n}, k)$ be a regular local ring of dimension $h \geq 2$, and I an ideal in R such that R/I is almost stretched Artinian and Gorenstein. If $\mathbf{k}^{1/2} \subseteq \mathbf{k}$, then we can find integers $s \geq t+1 \geq 3$, a minimal system of generators $x_1, \dots, x_h$ of $\mathfrak{n}$ and an element $a \in R$, such that I is generated by*

$$\{x_1x_j\}_{j=3,\dots,h} \quad \{x_ix_j\}_{2\leq i<j\leq h} \quad \{x_j^2 - x_1^s\}_{j=3,\dots,h} \quad x_2^2 - ax_1x_2 - x_1^{s-t+1}, \quad x_1^t x_2.$$

Proof. We know that integers $s \geq t+1 \geq 3$ can be found and a minimal system of generators $y_1, \dots, y_h$ of $\mathfrak{n}$ can be constructed in such a way that I is generated by

$$\{y_1y_j\}_{j=3,\dots,h} \quad \{y_iy_j\}_{2\leq i<j\leq h} \quad \{y_j^2 - u_jy_1^s\}_{j=3,\dots,h} \quad y_2^2 - by_1y_2 - wy_1^{s-t+1}, \quad y_1^t y_2.$$

with $w, u_3, \dots, u_h \notin \mathfrak{n}$ and $b \in R$. By Lemma 3.6 we can find elements $v, r_3, \dots, r_h \in R$ such that modulo I we have

$$v^2 \cong (1/w), \quad r_3^2 \cong (1/u_3), \dots, r_h^2 \cong (1/u_h).$$

From this is clear that $v, r_3, \dots, r_h$ are units in R and we can make the following change of minimal generators for $\mathfrak{n}$:

$$x_1 = y_1, \quad x_2 = vy_2, \quad x_3 = r_3y_3, \quad \dots, \quad x_h = r_hy_h.$$

We have

$$y_2^2 - by_1y_2 - wy_1^{s-t+1} = (x_2^2/v^2) - bx_1(x_2/v) - wx_1^{s-t+1} \in I,$$

hence $x_2^2 - bvx_1x_2 - v^2wx_1^{s-t+1} \in I$. Since $v^2w = 1 + d$ with $d \in I$, if we let $a := bv$, we get

$$x_2^2 - ax_1x_2 - x_1^{s-t+1} \in I.$$

Further for every $j = 3, \dots, h$ we have

$$y_j^2 - u_jy_1^s = (x_j/r_j)^2 - u_jx_1^s \in I,$$

hence $x_j^2 - r_j^2u_jx_1^s \in I$. Since $r_j^2u_j = 1 + e$ with $e \in I$, we get for every $j = 3, \dots, h$

$$x_j^2 - x_1^s \in I.$$

Hence I contains the ideal generated by

$$\{x_1x_j\}_{j=3,\dots,h} \quad \{x_ix_j\}_{2\leq i<j\leq h} \quad \{x_j^2 - x_1^s\}_{j=3,\dots,h} \quad x_2^2 - ax_1x_2 - x_1^{s-t+1}, x_1^t x_2.$$

Since by Corollary 4.6 these two ideals have the same Hilbert function, they coincide. $\square$

5 Classification of Gorenstein local algebras with Hilbert function (1,2,2,2,1,1,1)

We have seen in Section 3 that the Cohen-Macaulay type determines the moduli class of stretched Artinian local rings. In the case of almost stretched Artinian local rings, the problem is not so easy, even in the Gorenstein case. For example it has been proved in [1] that if A is Gorenstein with Hilbert function $1,2,2,1$, we have only two models, namely the ideals $I=(x^2,y^3)$ and $I=(xy,x^3-y^3)$. But already in the next case with symmetric Hilbert function $1,2,2,2,1$, we have at least three different models, namely two ideals which are homogeneous $I=(x^2,y^4)$, $I=(xy,x^4-y^4)$ and one which is not homogeneous, the ideal $I=(x^4+2x^3y,y^2-x^3)$.

But things become soon even more complicate, already in the complete intersection case, the case $h=2$. We are going to study the moduli problem for complete intersection local rings with Hilbert function $1,2,2,2,1,1,1$. We will see that in this case we have a one-dimensional family.

In the following, $(R,\mathfrak{n})$ is a two dimensional regular local ring such that $\mathbf{k}=R/\mathfrak{n}$ has the property $\mathbf{k}^{1/2}\subseteq\mathbf{k}$; I is an ideal in R such that $A=R/I$ is Gorenstein with Hilbert function $1,2,2,2,1,1,1$. We are not going into all the details, better we try simply to give an idea of what is going on.

By the main structure theorem we know that there exists a system of generators y_1,y_2 of $\mathfrak{n}$ and an element $a\in R$ such that, Proposition 4.8,

$$I=(y_1^3y_2,\,y_2^2-ay_1y_2-y_1^4).$$

Case 1: $a\notin\mathfrak{n}$. Let us change the generators as follows:

$$z_1=ay_1-y_2,\quad z_2=y_1^3+ay_2.$$

We have

$$d:=det\begin{pmatrix} a & y_1^2\\ -1 & a\end{pmatrix}=a^2+y_1^2\notin\mathfrak{n}$$

so that z_1,z_2 is a minimal system of generators of $\mathfrak{n}$. We have

$$z_1z_2=(ay_1-y_2)(y_1^3+ay_2)=-a(y_2^2-ay_1y_2-y_1^4)-y_1^3y_2\in I.$$

Since I contains the product of two minimal generators of $\mathfrak{n}$, then there exists a system of generators x,y of $\mathfrak{n}$ such that

$$I=(xy,y^4-x^6).$$

Case 2: $a\in\mathfrak{n}$. In this case, we write $a=by_1+cy_2$, and choose $v\in R$ such that $1-cy_1\cong v^2$ modulo I, Lemma 3.6. Notice that $v\notin\mathfrak{n}$, so that we can change the generators as follows

$$x_1=y_1,\quad x_2=vy_2$$

and prove that

$$I = (x_1^3 x_2, x_2^2 - d x_1^2 x_2 - x_1^4)$$

with $d = bv^{-1} \in R$.

Case 2a: $d \in \mathfrak{n}$. In this case we write $d = fx_1 + ex_2$ and choose $v \in R$ such that $v^2 \cong 1 - ex_1^2$ modulo I. It is clear that $v \notin \mathfrak{n}$ so that we can change the generators of $\mathfrak{n}$ by letting

$$x = x_1, \qquad y = vx_2.$$

Then it is easy to prove that

$$I = (x^3 y, y^2 - x^4).$$

Let now consider the case $d \notin \mathfrak{n}$. We distinguish two subcases, $d^2 + 4 \in \mathfrak{n}$ and $d^2 + 4 \notin \mathfrak{n}$. We first assume that

Case 2b1: $d^2 + 4 \in \mathfrak{n}$. In this case we have modulo I

$$(x_2 - (d/2)x_1^2)^2 \cong x_1^4 + (d^2/4)x_1^4 \cong x_1^4(1 + (d^2/4)) = ex_1^5$$

with $e \in R$. It follows that if we let

$$l := x_2 - (d/2)x_1^2 + (e/d)x_1^3 + (e^2/d^3)x_1^4$$

then $l^2 \in I$. Modulo I we have

$$x_1^3 l = x_1^3(x_2 - (d/2)x_1^2 + (e/d)x_1^3 + (e^2/d^3)x_1^4) \cong -(d/2)x_1^5 + (e/d)x_1^6 =$$

$$= x_1^5(-d/2 + (e/d)x_1) = vx_1^5$$

with $v \notin \mathfrak{n}$. It follows that $J = (l^2, x_1^3 l - vx_1^5) \subseteq I$. Next we prove $J = I$.

Notice that x, l form a minimal system of generators of $\mathfrak{n}$ and we denote by L the initial form of l in the associated graded ring $gr_{\mathfrak{n}}(R)$. In order to prove that $I = J$ we need to show that the Hilbert function of R/J is $1, 2, 2, 2, 1, 1, 1$. We have

$$(X^3 L, L^2) \subseteq J^* \subseteq I^*,$$

so we have to prove that

$$Z^7 \in J.$$

Notice that, modulo J, we have

$$vx^7 = x^5 l = v^{-1}(x^3 l^2) = 0.$$

Hence $x^7 \in J$, so $(l^2, x_1^3 l - vx_1^5) = I$.

Now we have

$$(l^2, x_1^3 l - vx_1^5) = (l^2, (x_1^3 l/v) - x_1^5) = ((l/v)^2, x_1^3(l/v) - x_1^5).$$

If we let $x := l/v, y = x_1$ then $\mathfrak{n} = (x, y)$ and

$$I = (x^2, xy^3 - y^5).$$

Case 2b2: $d^2 + 4 \notin \mathfrak{n}$. We can find $c, e \in R \setminus \mathfrak{n}$ such that modulo I we have $c^2 \cong d^2 + 4$ and $e^2 \cong -(2/c)$, Lemma 3.6. We let $p := d/c$ and change the generators of $\mathfrak{n}$ by letting

$$x = (x_1/e), \qquad y = x_2 + p(x_1/e)^2.$$

We get

$$x_1 = xe, \qquad x_2 = y - px^2$$

so that modulo I we get

$$0 \cong x_1^3 x_2 = x^3 e^3 (y - px^2) = e^3 (x^3 y - px^5)$$

which implies $x^3 y - px^5 \in I$. Further

$$0 \cong x_2^2 - dx_1^2 x_2 - x_1^4 = (y - px^2)^2 - dx^2 e^2 (y - px^2) - x^4 e^4 =$$

$$= y^2 - x^2 y(2p + de^2) + x^4 (p^2 + de^2 p - e^4) \cong y^2 - x^4$$

because

$$2p + de^2 = 2(d/c) + de^2 \cong 2(d/c) - 2(d/c) = 0$$

and

$$p^2 + de^2 p - e^4 = (d^2/c^2) + (d/c)d(-2/c) - (4/c^2) = -(d/c)^2 - (2/c)^2 \cong -1.$$

This proves that $J := (x^3 y - px^5, y^2 - x^4) \subseteq I$. We remark that

$$p^2 - 1 = (d/c)^2 - 1 = (d^2 - c^2)/c^2 \cong -(2/c)^2,$$

and this implies

$$p^2 - 1 \notin \mathfrak{n}.$$

In order to prove that $I = J$ we need to show that the Hilbert function of R/J is $1, 2, 2, 2, 1, 1, 1$. We have

$$(X^3 Y, Y^2) \subseteq J^* \subseteq I^*.$$

Further

$$y(x^3 y - px^5) - x^3 (y^2 - x^4) = -pyx^5 + x^7 \in J$$

which implies $x^5 y - (1/p)x^7 \in J$. Thus we have

$$x^2 (x^3 y - px^5) - (x^5 y - (1/p)x^7) = \frac{1 - p^2}{p} x^7 \in J.$$

From this we get $x^7 \in J$, hence

$$(X^3Y, Y^2, X^7) \subseteq J^* \subseteq I^*.$$

These ideals have the same Hilbert function so that we finally get

$$I = (x^3y - px^5, y^2 - x^4)$$

with

$$p \notin \mathfrak{n}, \qquad p^2 - 1 \notin \mathfrak{n}.$$

We have thus found three models (Case 1, Case 2a, Case 2b1) and a one dimensional family, Case 2b2. We summarize the models in the following table

Case 1	$I = (xy, y^4 - x^6)$	
Case 2a	$I = (x^3y, y^2 - x^4)$	
Case 2b1	$I = (x^2, xy^3 - y^5)$	
Case 2b2	$I = (x^3y - px^5, y^2 - x^4)$	$p \notin \mathfrak{n}$ and $p^2 - 1 \notin \mathfrak{n}$

At this point a natural question is whether we can pass from a model to another by a changing of generators of $\mathfrak{n}$.

For example, the model $I = (xy, y^4 - x^6)$ of Case 1 cannot be reached by any of the other models, because it is quite easy to see that, however we choose the element $a \in \mathfrak{n}$, the ideal $(x^3y, y^2 - axy - x^4)$ does not contain the product of two minimal generators of the maximal ideal $\mathfrak{n}$.

We are able to prove that all the models we have found are indeed non isomorphic, but here we give a proof only for the ideals in the family of Case 2b2.

Proposition 5.1. *Let $p, q \in R$ such that $p, q, p^2-1, q^2-1 \notin \mathfrak{n}$. If $\mathfrak{n} = (x, y) = (z, v)$ and $(x^3y - px^5, y^2 - x^4) = (z^3v - qz^5, v^2 - z^4)$ then $p^2 - q^2 \in \mathfrak{n}$.*

Proof. Let $I := (x^3y - px^5, y^2 - x^4)$; we will use the equalities $(\mathfrak{n}/I)^3 = (\overline{x}^3, \overline{x}^2\overline{y})$, $(\mathfrak{n}/I)^4 = (\overline{x}^4)$, $(\mathfrak{n}/I)^5 = (\overline{x}^5)$.

We first use the generators $v^2 - z^4$ to get $v^2 \in \mathfrak{n}^4 + I \subseteq (y, x^4)$. This implies $v \in (y, x^2)$ so that $v = ex^2 + by$, with $b \notin \mathfrak{n}$. Since modulo I we have

$$v^2 = e^2x^4 + 2ebx^2y + b^2y^2 \cong e^2x^4 + 2ebx^2y + b^2x^4,$$

we get $2ebx^2y \in \mathfrak{n}^4 + I$ which gives $e \in \mathfrak{n}$ and finally

$$v = ax^3 + by$$

with $a \in R$, $b \notin \mathfrak{n}$. We also have $z = cx + dy$ with

$$det \begin{pmatrix} ax^2 & c \\ b & d \end{pmatrix} = adx^2 - bc \notin \mathfrak{n}$$

which implies $c \notin \mathfrak{n}$.

Now, modulo I, we have $0 \cong v^2 - z^4 = b^2x^4 - c^4x^4 + t$ with $t \in \mathfrak{n}^5$ which implies $b^2 - c^4 \in \mathfrak{n}$. We also have

$$0 \cong z^3v - qz^5 = z^3(v - qz^2) \cong c^3bpx^5 - qc^5x^5 + f$$

with $f \in \mathfrak{n}^6$. This implies $c^3bp - qc^5 \in \mathfrak{n}$, hence $bp - qc^2 \in \mathfrak{n}$. Since $b^2 - c^4 \in \mathfrak{n}$ we easily get the conclusion $p^2 - q^2 \in \mathfrak{n}$. $\square$

With the methods explained before we can manage also the case with Hilbert function $1, 3, 2, 1$. This case was the unique left case in order to classify, up to isomorphism, Artinian Gorenstein $\mathbf{k}$-algebras of degree 7. Thus we can solve Question 4.4. of [1]. We prove that if R/I is Gorenstein with Hilbert function $1, 3, 2, 1$, then, after a possible change of generators of $\mathfrak{n}$, either

$$I = (xy, xz, yz, x^3 - y^3, z^2 - y^3) \quad \text{or} \quad I = (x^3, y^2, yz, xz, z^2 - x^2y).$$

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